IDEMPOTENT IDEALS AND NON-FINITELY GENERATED PROJECTIVE MODULES OVER INTEGRAL GROUP RINGS OF POLYCYCLIC-BY-FINITE GROUPS

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ABSTRACT. We prove that every non-finitely generated projective module over the integral group ring of a polycyclic-by-finite group G is free if and only if G is polycyclic.

1. INTRODUCTION

A question about existence of idempotent ideals is often embroidered in a definition of certain classes of rings. For example (see [14, Prop. 6.3]), a hereditary noetherian prime ring R is Dedekind prime, if R has no nontrivial idempotent ideals. However, it is usually difficult to decide whether a ring in a given class of rings has a nontrivial idempotent ideal.

For instance, Akasaki [1, Thm. 2.1] proved that the integral group ring of any finite non-soluble group has a nontrivial idempotent ideal. This result was completed by Roggenkamp [18]: the integral group ring of a finite soluble group has no nontrivial idempotent ideals. Later P. Smith [20, Thm. 2.2] extended Roggenkamp's result to integral group rings of polycyclic groups. In fact, using arguments similar to Akasaki's, it is not difficult to conclude (see Proposition 4.5) that the integral group ring of a polycyclic-by-finite group G has no nontrivial idempotent ideals if and only if G is polycyclic (i.e., if G is soluble).

By Bass [4], it is quite often that non-finitely generated projective modules over a given noetherian ring are free. For instance (see [4, Cor. 4.5]), this is true for modules over an indecomposable commutative noetherian ring. Confirming this observation, Swan [25, Thm. 7] proved that every non-finitely generated projective module over the integral group ring of a finite soluble group is free. Unfortunately, contrary to what was expected (see Bass [4, p. 24]), some infinitely generated non-free modules over integral group rings of non-soluble finite groups were found by Akasaki [2] and Linnell [10]. Namely, they proved that, if G is a finite non-soluble

group, then there exists an infinitely generated projective $\mathbb{Z}G$-module which is not free.

Moreover, Akasaki [2] revealed the role played by idempotent ideals in those kind of results. The main idea was to use an idempotent ideal I of $\mathbb{Z}G$ and a remarkable theorem of Whitehead [26] to guarantee the existence of a projective $\mathbb{Z}G$-module P whose trace is equal to I. The proof was completed by the following observation: every finitely generated projective module over the integral group ring of a finite group is a generator (hence P has no finitely generated direct summands).

In this paper we extend this Akasaki and Linnell result to the class of (infinite) polycyclic-by-finite groups. Namely (see Theorem 6.6), we prove that every non-finitely generated projective module over the integral group ring of a polycyclic-by-finite group G is free if and only if G is polycyclic (that is, if G is soluble). Moreover, if G is not polycyclic, then there exists a projective $\mathbb{Z}G$-module without finitely generated direct summands.

One implication of this theorem (that every non-finitely generated projective module over the integral group ring of a polycyclic group is free) was proved by the third author [20, Thm. 2.3]. However the proof was a little brief, so we have included a full proof of this result using [21, Lemma 4.6], which is a criterion for a projective module to be finitely generated.

We also give another proof of this implication based on Cohn's universal localizations. Though it is longer, it may be used to extend the results of the paper to the case of soluble groups.

To prove the other implication, extending Akasaki's result we show (in Proposition 5.7) that every finitely generated projective module over the integral group ring of a polycyclic-by-finite group is a generator. As a key preliminary step, using K-theory, we prove (in Corollary 5.5) that, if G is a polycyclic-by-finite group and P is a finitely generated projective $\mathbb{Z}G$-module, then $(P \otimes_{\mathbb{Z}} \mathbb{C})^n$ is a free $\mathbb{C}G$-module for some n.

Despite the existence of 'bad' projective modules over some integral group rings (say, over $\mathbb{Z}A_5$) being established, the structure of these modules is widely unknown. For instance, the old question of P. Linnell (Problem 8.34 in [8]) about the existence of a non-finitely generated indecomposable projective module over the integral group ring of a finite group is still open.

To make the paper self-contained and available even to neophytes, we included most basic definitions and explanations from group theory and the theory of projectives modules.

2. Projective modules

All rings will be associative rings with unity 1, and most modules will be (unitary) right modules.

Recall that a module F over a ring R is *free*, if it is isomorphic to a direct sum of copies of R, that is, $F \cong R^{(\alpha)}$ for some cardinal α. A module P is said to be *projective*, if P is isomorphic to a direct summand of a free module. For instance, if $f \in R$ is an idempotent, then $R_R = fR \oplus (1 - f)R$, hence fR is a projective (right) R-module. Every finitely generated projective module P over a ring R is isomorphic to the module ER^n, where E is an idempotent $n \times n$ matrix over R, and R^n is a column of height n over R.

The trace of P, $\mathrm{Tr}(P)$, is defined as the sum of images of all morphisms from P to R_R. For instance, if $f \in R$ is an idempotent and $P = fR$, then $\mathrm{Tr}(P) = RfR$ is the two-sided ideal of R generated by f. More generally, if a finitely generated projective module P is given by an idempotent matrix E as above, then $\mathrm{Tr}(P)$ is the two-sided ideal of R generated by the entries of E.

Fact 2.1. *(see* [9, Prop. 2.40]*) If P is a projective R-module, then $I = \mathrm{Tr}(P)$ is an idempotent ideal. Moreover, $P = PI$, and I is the least two-sided ideal of R with this property.*

Thus, the trace of any projective module is an idempotent ideal. However, the question whether a given idempotent ideal is the trace of some projective module is hard to answer. For example, if R is a commutative ring and I is a finitely generated idempotent ideal of R, then (see [9, Lemma 2.43]) I is generated by an idempotent f, hence $I = \mathrm{Tr}(fR)$.

If R is not commutative, and I is an idempotent ideal of R finitely generated on one side, it is not always true that I is generated by an idempotent. But such an I is a trace of a projective module as the following fact shows.

Fact 2.2. [26, Cor. 2.2] *Let I be a two-sided idempotent ideal of a ring R such that I is finitely generated as a left ideal. Then there exists a projective right R-module P such that $\mathrm{Tr}(P) = I$.*

Following Bass, we say that a projective module P over a ring R is ω-*big* (uniformly ω-big in the terminology of Bass) if, for every two-sided ideal I of R, the (projective) R/I-module P/PI is not finitely generated. In particular, P itself is not finitely generated.

If R is a ring, $\mathrm{Jac}(R)$ will denote the Jacobson radical of R. The following fact shows that ω-big projective modules are often free.

Fact 2.3. [4, Thm. 3.1] *Let R be a ring such that $R/\mathrm{Jac}(R)$ is right noetherian. Then every ω-big countably generated projective right R-module is free.*

A nonzero (projective) module P over a ring R is said to be *stably free* (of rank m), if $P \oplus R^n \cong R^{n+m}$ for some n, m. If R is right noetherian, then m is the ratio of Goldie dimensions of P and R_R.

We say that P is a *generator* if, for some k, there is an epimorphism $P^k \to R_R$. Clearly this is the same as $\mathrm{Tr}(P) = R$. It is easily checked (as T. Stafford pointed out to the second author) that every stably free module over a (right) noetherian ring is a generator. We need the following refinement of this fact.

Fact 2.4. [14, Thm. 11.1.3] *Let P be a stably free module over a right noetherian ring of Krull dimension h. If the rank of P exceeds h, then P is free.*

The following theorem of Kaplansky imposes severe restrictions on the size of indecomposable projective modules.

Fact 2.5. [3, Cor. 2.6.2] *Every projective module is a direct sum of countably generated modules.*

A ring R is said to be *local*, if R has a unique maximal right (and left) ideal, that is, if $R/\mathrm{Jac}(R)$ is a skew field.

Fact 2.6. [3, Cor. 26.7] *Every projective module over a local ring is free.*

We also need the following result on projective modules.

Fact 2.7. [11, Lemma 4] *Let R be a subring of the ring S and let P be a projective R-module. If the induced (projective) S-module $P \otimes_R S$ is finitely generated, then P is finitely generated.*

3. Groups

We will use e for the unity of a group G, and then $\{e\}$ will denote the subgroup of G consisting of e.

A series of groups $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n$ is said to be a *subnormal series* of a group G, if $G_n = G$, and each G_i is a normal subgroup of G_{i+1}, $i = 0, \ldots, n-1$.

A group G is *soluble*, if it has a subnormal series with abelian factors G_{i+1}/G_i. The *commutant* $G' = [G, G]$ of a group G is defined as the (normal) subgroup of G generated by all commutators $[a, b] = aba^{-1}b^{-1}$, $a, b \in G$. Analogously, setting $G^{(n)} = (G^{(n-1)})'$, one can define the *nth commutant* of G, $G^{(n)}$. It is easily checked that each $G^{(n)}$ is a normal subgroup of G.

Fact 3.1. *A group G is soluble if and only if $G^{(n)} = \{e\}$ for some n.*

A group G is said to be *perfect*, if $G = [G, G]$, that is, G coincides with its commutant. For instance, every alternating group A_n, $n \geq 5$ is perfect. Thus, if G is a finite non-soluble group, then some commutant of G is a nonidentity perfect normal subgroup.

A group G is called *polycyclic*, if G has a subnormal series with cyclic factors. Since every cyclic group is abelian, every polycyclic group is soluble. If all factors in a subnormal series of G are either cyclic or finite, G is said to be a *polycyclic-by-finite* group. For instance, every polycyclic group is polycyclic-by-finite. The

number of infinite cyclic factors in a subnormal series of a polycyclic-by-finite group G is called *the Hirsch number* of G, $h(G)$.

Fact 3.2. *A polycyclic-by-finite group is soluble if and only if it is polycyclic.*

If G is a polycyclic-by-finite group, then the factors in a subnormal series of G can be rearranged such that all of them but the last are cyclic. Thus every polycyclic-by-finite group G has a normal polycyclic subgroup of finite index.

Now we extend a standard characterization of non-soluble finite groups to general soluble-by-finite groups. This characterization should be known, but we were not able to find a precise reference.

Lemma 3.3. *Let G be a soluble-by-finite group which is not soluble. Then G has a nonidentity perfect normal subgroup.*

Proof. First recall that, if H, K and L are normal subgroups of (any) group G, then $[HK, L] = [H, L] \cdot [K, L]$. Indeed, the inclusion $\supseteq$ is clear, and $\subseteq$ is easily checked using the following commutator identity: $[hk, l] = h[k, l]h^{-1} \cdot [h, l]$.

Now the claim of the lemma is trivial, if G is abelian, and also true (see after Fact 3.1), if G is finite.

Otherwise, let G_1 be a normal soluble subgroup of finite index in G, and let A be the last nonidentity term in the derived series of G_1. Thus A is a nontrivial normal abelian subgroup of G. Arguing by induction (on the length of the derived series of G_1), we may assume that the result has been already proved for G/A, that is, G/A has a nonidentity perfect normal subgroup H/A.

Thus $A \subset H$ are normal subgroups of G such that H is perfect modulo A, hence $H = H^{(n)}A$ for every n. We prove that H' is a perfect normal subgroup of G. Then $H = H'A$ would imply that $H' \neq \{e\}$.

Indeed, since $H = H^{(2)}A$, by the above remark we obtain $[H, A] = [H^{(2)}A, A] = [H^{(2)}, A] \cdot [A, A] = [H^{(2)}, A] \subseteq H^{(2)}$. Then, using the same remark, $H' = [H, H] = [H'A, H'A] = [H', H'A] \cdot [A, H'A] = H^{(2)}[H', A] \cdot [A, H'] \cdot [A, A] = H^{(2)} \cdot [H', A] \subseteq H^{(2)}$, hence H' is perfect. $\square$

It is known that every polycyclic-by-finite group G is residually finite, but we need a stronger version of this result. A group G is said to be *conjugacy separable*, if for any $g, h \in G$ such that g is not a conjugate of h in G, there is a normal subgroup H of finite index in G such that $\bar{g}$ is not a conjugate of $\bar{h}$ in G/H.

Fact 3.4. [17] *Every polycyclic-by-finite group G is conjugacy separable. Moreover, if $\gamma_1, \ldots, \gamma_k$ are different conjugacy classes of G, then there exists a normal subgroup H of finite index in G such that $\bar{\gamma}_1, \ldots, \bar{\gamma}_k$ are different conjugacy classes in G/H, where $\bar{\gamma}_i$ indicates the image of γ_i in G/H.*

4. INTEGRAL GROUP RINGS

If G is a group, then $\mathbb{Z}G$ will denote the integral group ring of G. Thus, every element $r \in \mathbb{Z}G$ has a unique representation $r = \sum_{i=1}^{k} n_i g_i$, where $n_i \in \mathbb{Z}$ and $g_i \in G$. In particular $1 \cdot e$ is the unity of $\mathbb{Z}G$ and we will usually write 1 instead. There is a natural epimorphism (the augmentation map) $\varepsilon \colon \mathbb{Z}G \to \mathbb{Z}$ given by $\varepsilon(r) = \sum_{i=1}^{k} n_i$. The kernel of this map is the *augmentation ideal* $\Delta(G)$ of G. Thus $r \in \Delta(G)$ if and only if $\sum_{i=1}^{k} n_i = 0$. It is easily seen that $\Delta(G)$ is the free abelian group generated by $\{g - 1 \mid g \in G\}$. Furthermore ε extends to an epimorphism of $n \times n$ matrices $\mathrm{M}_n(\mathbb{Z}G) \to \mathrm{M}_n(\mathbb{Z})$ by applying ε to each matrix entry.

If H is a subgroup of G, the inclusion $H \subseteq G$ induces an inclusion $\mathbb{Z}H \subseteq \mathbb{Z}G$. Let $\omega(H)$ denote the two-sided ideal of $\mathbb{Z}G$ generated by $\Delta(H)$. If H is normal in G, it is easily checked that $\omega(H) = \mathbb{Z}G \cdot \Delta(H) = \Delta(H) \cdot \mathbb{Z}G$. A natural epimorphism $G \to G/H$ induces an epimorphism of rings $\pi \colon \mathbb{Z}G \to \mathbb{Z}(G/H)$. The following fact describes the kernel of π.

Fact 4.1. *(see* [15, Lemma 1.8]*) If H is a normal subgroup of a group G, then we have the following exact sequence: $0 \to \omega(H) \to \mathbb{Z}G \xrightarrow{\pi} \mathbb{Z}(G/H) \to 0$. In particular, if $H = G$, we obtain $0 \to \Delta(G) \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$.*

We recall some basic facts on integral group rings of polycyclic-by-finite groups.

Fact 4.2. [14, Thm. 5.12] *If G is a polycyclic-by-finite group, then $\mathbb{Z}G$ is a noetherian ring.*

We also need the following powerful result of Roseblade.

Fact 4.3. [19, Cor. C3] *Let G be a polycyclic-by-finite group, and let M be a maximal two-sided ideal of $\mathbb{Z}G$. Then there exists a prime p and a normal subgroup H of finite index in G such that $p\mathbb{Z}$, $\omega(H) \subseteq M$.*

The proof of the following lemma is similar to Akasaki [1, Thm. 2.1].

Lemma 4.4. *Let G be a polycyclic-by-finite group which is not polycyclic. Then the integral group ring of G has a nontrivial idempotent ideal.*

Proof. Since G is not polycyclic, it is not soluble (see Fact 3.2). Thus, by Fact 3.3, G has a nonidentity normal perfect subgroup H, that is, $H = [H, H]$.

First we check that the augmentation ideal $\Delta(H)$ of $\mathbb{Z}H$ is idempotent. Indeed, as we have already noticed, the elements $h - 1$, $h \in H$ generate $\Delta(H)$ as an abelian group. Thus, it suffices to check that $h - 1 \in \Delta(H)^2$. If $h = [a, b] = aba^{-1}b^{-1}$ for some $a, b \in H$, then $h - 1 = ab(a^{-1}b^{-1} - b^{-1}a^{-1})$ Since $cd - dc = (c - 1)(d - 1) - (d - 1)(c - 1) \in \Delta(H)^2$ for all $c, d \in H$, we conclude that $h - 1 \in \Delta(H)^2$.

Suppose that h is an arbitrary element of H. Then $h = h_1 \cdot \ldots \cdot h_k$, where $h_i = [a_i, b_i]$ for some $a_i, b_i \in H$. Using an induction on k, we may assume that

$k \geq 2$ and $h' - 1 \in \Delta(H)^2$, where $h' = h_2 \cdot \ldots \cdot h_k \in H$. Then $h - 1 = (h_1 - 1)(h' - 1) + (h_1 - 1) + (h' - 1) \in \Delta(H)^2$ by what we have already proved.

Now, as we have previously noticed, $\omega(H) = \mathbb{Z}G \cdot \Delta(H) = \Delta(H) \cdot \mathbb{Z}G$. Since $\Delta(H)^2 = \Delta(H)$, one easily derives that $\omega(H)$ is a nonzero idempotent two-sided ideal of $\mathbb{Z}G$. From the obvious inclusion $\omega(H) \subseteq \Delta(G)$ it follows that $\omega(H) \neq \mathbb{Z}G$. $\qquad\square$

Now we are in a position to characterize integral group rings of polycyclic-by-finite groups possessing a nontrivial idempotent ideal.

Proposition 4.5. *Let G be a polycyclic-by-finite group. Then the following are equivalent:*

  1) $\mathbb{Z}G$ has no nontrivial idempotent two-sided ideals.

  2) G is polycyclic.

Proof. 1) $\Rightarrow$ 2) follows from Lemma 4.4, and 2) $\Rightarrow$ 1) is [20, Thm. 2.2.]. $\qquad\square$

5. One implication

Recall that $K_0(R)$ denotes the Grothendieck group of finitely generated projective modules over a ring R, that is, an abelian group whose generators are finitely generated projective R-modules subject to relations $[P] = [Q] + [Q']$ for all short exact (hence split) sequences $0 \to Q \to P \to Q' \to 0$. The Grothendieck group $G_0(R)$ of R is a similar abelian group defined on the set of finitely generated R-modules.

Note that (see [9, Cor. 5.60]), if R is a two-sided noetherian ring, then the left and right global dimensions of R coincide and can be calculated as the supremum of projective dimensions of cyclic R-modules. If this supremum is finite, R is said to be a *regular* ring.

Fact 5.1. [16, Prop. 3.8] *Let R be a noetherian regular ring. Then the natural map $c: K_0(R) \to G_0(R)$ sending $[P]$ to $[P]$ is an isomorphism called the* Cartan *map.*

In general the Cartan map will not be mono or epi. What is important for us is the following instance of Fact 5.1.

Fact 5.2. [15, Thm. 3.13] *If G is a polycyclic-by-finite group, then the ring $\mathbb{C}G$ is regular. Thus $K_0(\mathbb{C}G)$ is canonically isomorphic to $G_0(\mathbb{C}G)$.*

If $R \subseteq S$ are rings and P is a finitely generated R-module, it induces a finitely generated S-module $P \otimes_R S$. If S is flat as a left R-module, this gives rise to the induction map $G_0(R) \to G_0(S)$, and similarly for K_0. For instance, if H is a subgroup of a group G, and S is a ring, then induction via the inclusion $SH \subseteq SG$ induces maps $G_0(SH) \to G_0(SG)$ and $K_0(SH) \to K_0(SG)$.

The following fact is known as J. Moody's induction theorem.

Fact 5.3. *(see* [16, Thm. 3.6.9]*) Let S be a noetherian ring and let G be a polycyclic-by-finite group. Suppose that $G_1, \ldots, G_k$ are representatives of the conjugacy classes of the maximal finite subgroups of G. Then $G_0(SG)$ is generated by the images of various $G_0(SG_i)$ under the induction map.*

If R is a ring, then $H_0(R) = R/[R, R]$ will denote the factor of R by the additive subgroup generated by additive commutators $[r, s] = rs - sr$ for $r, s \in R$. If $R = SG$ and S is commutative, then $[R, R]$ is an S-submodule of R, and $H_0(R)$ is isomorphic to the free S-module $\text{class}_0(G)$ generated by the set $\text{con}(G)$ of conjugacy classes of G.

The Hattori–Stallings map, HS, sends finitely generated projective R-modules to elements of $H_0(R)$ in the following way. Let P be a finitely generated projective R-module, hence $P \cong ER^n$ for some idempotent $n \times n$ matrix E over R. Then the trace of E is an element $r \in R = SG$, and $HS(P)$ is the image $\bar{r}$ of r in $H_0(R) = R/[R, R]$ (see [22] for more on that). For example, if S is a field, then $HS(P)$ is an element of the S-vector space $\text{class}_0(G)$ with $\text{con}(G)$ as a basis.

The Hattori–Stallings map induces a homomorphism of abelian groups $HS \colon K_0(R) \to H_0(R)$ (note that, even if S is commutative, $K_0(R)$ carries no natural S-module structure). For instance, the image of $[R] \in K_0(R)$ is $\bar{1} = 1 \cdot \{e\}$.

Lemma 5.4. *If G is a polycyclic-by-finite group, then the map $K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \to \text{class}_0(G)$ induced by the Hattori–Stallings map is an embedding.*

Proof. By [12, Lemma 2.15] we have the following commutative diagram:

$$
\begin{array}{ccc}
\varinjlim K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\ f\ } & K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
{\scriptstyle h} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle HS} \\
\text{class}_0(G)_f & \xrightarrow{\ g\ } & \text{class}_0(G)
\end{array}
$$

Here the direct limit is taken over the set of all finite subgroups H of G with respect to inclusion and G-conjugation (see [5, p. 91]). The morphism f is induced by induction maps $K_0(\mathbb{C}H) \to K_0(\mathbb{C}G)$. Furthermore, $\text{class}_0(G)_f$ denotes the $\mathbb{C}$ vector space spanned by the conjugacy classes of G whose elements have finite order, and g is a natural inclusion.

As is explained in [12, Thm. 2.22] (or alternatively use [5, Prop. of §1.8]), J. Moody's induction theorem (Fact 5.3) and Fact 5.2 imply that f is an isomorphism. Since g is an embedding, the same is true for HS. $\qquad \square$

By Swan [24, Thm 8.1], if P is a finitely generated projective module over the integral group ring of a finite group G, then $P \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free $\mathbb{Q}G$-module. Thus the following is a weak form of this result.

Lemma 5.5. *Let P be a finitely generated projective module over the integral group ring of a polycyclic-by-finite group G. Then $(P \otimes_{\mathbb{Z}} \mathbb{C})^n$ is a stably free $\mathbb{C}G$-module for some n.*

Proof. Let $P' = P \otimes_{\mathbb{Z}} \mathbb{C}$. Clearly P' is a finitely generated projective $\mathbb{C}G$-module, that is $[P'] \in K_0(\mathbb{C}G)$. The kernel of the natural map $K_0(\mathbb{C}G) \to K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C}$ is the torsion part of $K_0(\mathbb{C}G)$. Thus, by Lemma 5.4, it suffices to prove that $HS(P') \in \mathbb{Z}\bar{1}$.

Let $HS(P') = \sum_{i=1}^{k} \alpha_i \gamma_i$, where $0 \neq \alpha_i \in \mathbb{C}$ and γ_i are different conjugacy classes of G. If $HS(P') \notin \mathbb{Z}\bar{1}$, then there is a conjugacy class $\gamma_l \neq \{e\}$ (where $1 \leq l \leq k$). By Fact 3.4, there is a normal subgroup H of finite index in G such that $\bar{\gamma}_1, \ldots, \bar{\gamma}_n$ are different conjugacy classes of G/H, in particular $\bar{\gamma}_l \neq \{\bar{e}\}$.

Now $P'/P'\omega(H)$ is a finitely generated projective $\mathbb{C}(G/H)$-module (see Fact 4.1), and clearly $P'/P'\omega(H) \cong P/P\omega(H) \otimes_{\mathbb{Z}} \mathbb{C}$. Furthermore, $P/P\omega(H)$ is a finitely generated projective $\mathbb{Z}(G/H)$-module. Then, by the aforementioned Swan's result, $P/P\omega(H) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free $\mathbb{Q}[G/H]$-module, hence $P'/P'\omega(H)$ is a free $\mathbb{C}(G/H)$-module. But $HS(P'/P'\omega(H)) = \sum_{i=1}^{k} \alpha_i \bar{\gamma}_i$ has a nonzero coefficient by $\bar{\gamma}_l \neq \{\bar{e}\}$, a contradiction. $\square$

Corollary 5.6. *Let P be a finitely generated projective module over the integral group ring of a polycyclic-by-finite group G. Then $(P \otimes_{\mathbb{Z}} \mathbb{C})^n$ is a free $\mathbb{C}G$-module for some positive integer n.*

Proof. By Lemma 5.5, $Q = (P \otimes_{\mathbb{Z}} \mathbb{C})^n$ is a stably free $\mathbb{C}G$-module for some n, and we may assume that $n \geq mh + 1$, where h is the Hirsch number of G and m is the Goldie dimension of $\mathbb{C}G$.

Then the Goldie dimension of Q is at least $mh + 1$, hence the rank of Q (which is the ratio of Goldie dimensions of Q and $\mathbb{C}G$) exceeds h. Also, by [14, Prop. 6.1.1], the Krull dimension of $\mathbb{C}G$ is equal to h. It remains to apply Fact 2.4. $\square$

By Akasaki [2, Cor. 1.4] every finitely generated projective module over the integral group ring of a finite group is a generator. We generalize this result as follows.

Proposition 5.7. *Let P be a finitely generated projective module over the integral group ring of a polycyclic-by-finite group G. Then P is a generator.*

Proof. Otherwise $I = \text{Tr}(P)$ is a nontrivial two-sided ideal of $\mathbb{Z}G$ such that $P = PI$. If M is any maximal ideal of $\mathbb{Z}G$ containing $\text{Tr}(P)$, then $P = PM$. By Fact 4.3, there is a normal subgroup H of finite index in G such that $\omega(H) \subseteq M$. Write $\overline{M} = M/\omega(H)$, the image of M in $\mathbb{Z}G/\omega(H) \cong \mathbb{Z}(G/H)$. Then $P' = P/P\omega(H)$ is a finitely generated projective $\mathbb{Z}(G/H)$-module such that $P'\overline{M} = P'$. But G/H is a finite group, hence (by Akasaki's result) $P' = 0$ and we deduce that $(P \otimes_{\mathbb{Z}}$

$\mathbb{C})^n \omega(H) = (P \otimes_{\mathbb{Z}} \mathbb{C})^n$ for all positive integers n. Using Corollary 5.6, we conclude that $P \otimes_{\mathbb{Z}} \mathbb{C} = 0$, consequently $P = 0$ which is a contradiction. $\square$

Now we prove the first implication of the foregoing theorem.

Theorem 5.8. *Let G be a polycyclic-by-finite group which is not polycyclic. Then there exists a projective $\mathbb{Z}G$-module P such that P has no finitely generated direct summands. In particular, P is not free.*

Proof. By (the proof of) Lemma 4.4, there exists a nontrivial normal subgroup H of G such that $I = \omega(H)$ is an idempotent ideal. Since $\mathbb{Z}G$ is noetherian, by Fact 2.2 there exists a projective $\mathbb{Z}G$-module P such that $\mathrm{Tr}(P) = I$.

Suppose that Q is a finitely generated direct summand of P. By Proposition 5.7, $\mathrm{Tr}(Q) = \mathbb{Z}G$, hence $\mathrm{Tr}(P) = \mathbb{Z}G$, a contradiction. $\square$

6. The other implication

In this section we prove the following proposition.

Proposition 6.1. *Let P be a non-finitely generated projective module over the integral group ring of a polycyclic group G. Then P is free.*

But first we recall some definitions and results of [21]. Let I be a two-sided ideal of a ring R. We define a descending chain of two-sided ideals I^α as follows: put $I^{\alpha+1} = I^\alpha \cdot I$, and $I^\alpha = \bigcap_{\beta < \alpha} I^\beta$, if α is a limit ordinal. If γ is the least ordinal such that $I^\gamma = I^{\gamma+1}$, then denote I^γ by $\mathrm{k}^1(I)$.

Note that this definition is right handed: we obtain a left variant of it by setting $I^{\alpha+1} = I \cdot I^\alpha$ in the above definition. For every positive integer n define $\mathrm{k}^{n+1}(I) = \mathrm{k}^1(\mathrm{k}^n(I))$.

Let $\mathrm{id}(I)$ denote the sum of idempotent ideals of R contained in I. Then $\mathrm{id}(I)$ is the unique maximal idempotent ideal contained in I and $\mathrm{id}(I) \subseteq \mathrm{k}^n(I)$ for all n. A ring R is called *right shallow*, if for each ideal I of R there exists a positive integer n such that $\mathrm{k}^n(I) = \mathrm{id}(I)$.

Fact 6.2. [21, Cor. 2.10] *If G is a finite group, then the integral group ring of G is right and left shallow.*

Corollary 6.3. *Let G be a finite soluble group and let I be a proper two-sided ideal of $\mathbb{Z}G$. Then there exists a positive integer n such that $\mathrm{k}^n(I) = 0$.*

Proof. By Fact 6.2, $\mathrm{k}^n(I) = \mathrm{id}(I)$ for some n. But, by Roggenkamp [18, Prop.] the integral group ring of a finite soluble group has no nontrivial idempotent ideals. $\square$

Before proving the next lemma we need the following instance of B. Hartley's result.

Fact 6.4. [7, Thm. E] *If G is a finitely generated abelian group and $\Delta = \Delta(\mathbb{Z}G)$, then $\bigcap_n \Delta^n = 0$, in particular $\mathrm{k}^1(\Delta) = 0$.*

Lemma 6.5. *Let G be a polycyclic group. Then for every proper ideal I of $\mathbb{Z}G$ there exists a positive integer n such that $k^n(I) = 0$.*

Proof. Let M be a maximal ideal of $\mathbb{Z}G$ such that $I \subseteq M$. By Fact 4.3, there exists a normal subgroup H of finite index in G such that $\omega(H) \subseteq M$. Then $\bar{I} = I + \omega(H)/\omega(H)$ is a proper ideal of $\mathbb{Z}G/\omega(H) \cong \mathbb{Z}(G/H)$. By Corollary 6.3, there exists a positive integer m such that $k^m(\bar{I}) = \bar{0}$, that is, $k^m(I) \subseteq \omega(H)$. Thus it suffices to prove that $k^l(\omega(H)) = 0$ for some l. Moreover, since $\omega(H) = \mathbb{Z}G \cdot \Delta(H) = \Delta(H) \cdot \mathbb{Z}G$, it is enough to show that $k^l(\Delta(H)) = 0$.

We prove this by induction on the derived length of H. If H is abelian, the result follows from Fact 6.4. Otherwise $H' \neq \{e\}$ and we may assume that $k^s(\Delta(H')) = 0$ for some s, hence $k^s(\omega(H')) = 0$. Since H/H' is a finitely generated abelian group, $k^1(\Delta(H/H')) = 0$ by Fact 6.4. Thus $k^1(\Delta(H)) \subseteq \omega(H')$, therefore $k^{s+1}(\Delta(H)) \subseteq k^s(\omega(H')) = 0$. $\square$

Now we are ready to prove Proposition 6.1.

By Fact 2.5 we may assume that P is countably generated. Moreover, by Fact 2.3 it suffices to prove that P is ω-big, that is, P/PI is not finitely generated for every proper two-sided ideal I of $\mathbb{Z}G$.

Indeed, if P/PI is finitely generated then, by [21, Lemma 4.6], $P/P\,k^n(I)$ is finitely generated for every n. But, by Lemma 6.5, $k^n(I) = 0$ for some n, hence P is finitely generated, a contradiction.

Now we are in a position to formulate the main result of the paper.

Theorem 6.6. *Let G be a polycyclic-by-finite group. Then the following are equivalent:*

1) G is not polycyclic;

2) there exists a non-finitely generated projective $\mathbb{Z}G$-module that is not free;

3) there exists a projective $\mathbb{Z}G$-module P such that P has no finitely generated direct summands.

Proof. By Theorem 5.8 and Proposition 6.1. $\square$

7. Localizations

Let I be a prime ideal of a noetherian ring R. Then, by Goldie's theorem, R/I has a classical quotient ring $Q = Q(R/I)$ which is a simple artinian ring. Let Σ be the set of all square R-matrices that are regular modulo I (that is, become invertible over Q). Then Cohn's universal localization R_Σ can be defined as a ring homomorphism $R \to R_\Sigma$ such that 1) the images of all matrices in Σ are invertible over R_Σ, and 2) for every ring homomorphism $R \to S$ which inverts all matrices in Σ, there exists a unique ring homomorphism $R_\Sigma \to S$ completing the following commutative diagram:

$$R \longrightarrow R_\Sigma \dashrightarrow S$$
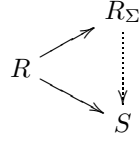
The ring R_Σ is unique (as a universal object) and can be constructed by adding formal inverses to matrices in Σ. For a more constructive way of building R_Σ see Malcolmson [13].

Fact 7.1. [6, Thm. 4.1] *The map $R \to R_\Sigma$ induces an embedding $R/I \to R_\Sigma/\operatorname{Jac}(R_\Sigma)$ such that $R_\Sigma/\operatorname{Jac}(R_\Sigma) \cong Q(R/I)$.*

Note that, if the prime ideal I is (right) localizable, then $R_\Sigma \cong R_I$, the localization of R at I.

If M is an R-module, we define the localization of M, M_Σ, setting $M_\Sigma = M \otimes_R R_\Sigma$. Thus M_Σ is an R_Σ-module.

A group G is said to be *poly-$\mathbb{Z}$*, if it has a subnormal series with factors isomorphic to $\mathbb{Z}$. Thus, every poly-$\mathbb{Z}$ group is polycyclic. Furthermore, every polycyclic group contains a normal poly-$\mathbb{Z}$ subgroup of finite index.

Fact 7.2. *(see [16, Lemma 3.7.8]) Every poly-$\mathbb{Z}$ group G is torsion-free and the integral group ring of G is an Ore domain.*

Lemma 7.3. *Let H be a poly-$\mathbb{Z}$ group, and let $f_A\colon \mathbb{Z}H^n \to \mathbb{Z}H^n$ be a homomorphism of (free) $\mathbb{Z}H$-modules given by left multiplication by an $n \times n$ matrix A over $\mathbb{Z}H$. If f_A is a monomorphism modulo $\Delta(H)$, then f_A is a monomorphism.*

Proof. By Strebel [23, Prop. 1.5], H is in the class $D(\mathbb{Z})$, that is, any map $f\colon P \to Q$ between projective $\mathbb{Z}H$-modules is a monomorphism, if the induced map of abelian groups $f \otimes 1\colon P \otimes_{\mathbb{Z}H} \mathbb{Z} \to Q \otimes_{\mathbb{Z}H} \mathbb{Z}$ is a monomorphism.

It remains to notice that $f_A \otimes 1$ is an endomorphism of a $\mathbb{Z}$-module $\mathbb{Z}^n \cong [\mathbb{Z}H/\Delta(H)]^n$ given by left multiplication by $\varepsilon(A)$. $\qquad\square$

We put in use the following result.

Fact 7.4. [25, Thm. 7] *Every non-finitely generated projective module over the integral group ring of a finite soluble group is free.*

Now we give a different proof of Proposition 6.1.

As above we may assume that P is countably generated, and we have to show that P/PI is not finitely generated for every proper two-sided ideal I of $\mathbb{Z}G$.

Suppose that P/PI is finitely generated. If M is a maximal two-sided ideal of $\mathbb{Z}G$ containing I, then P/PM is also finitely generated. We prove that this leads to a contradiction.

By Fact 4.3, there is a prime p and a normal subgroup H of finite index in G such that $p\mathbb{Z}$, $\omega(H) \subseteq M$. Moreover (see the remark above), we may assume that H is a poly-$\mathbb{Z}$ group.

Note that $P/P\omega(H)$ is a projective $\mathbb{Z}G/\omega(H) = \mathbb{Z}(G/H)$-module and G/H is a finite soluble group. If $P/P\omega(H)$ is not finitely generated, it is a free infinite rank $\mathbb{Z}(G/H)$-module by Fact 7.4. Since $\omega(H) \subseteq M$, we conclude that P/PM is a free infinite rank $\mathbb{Z}G/M$-module, a contradiction.

Thus $P/P\omega(H)$ is a finitely generated $\mathbb{Z}G$-module. Since H is of finite index in G and $P\omega(H) = P\Delta(H)$, we deduce that $P/P\Delta(H)$ is a finitely generated $\mathbb{Z}H$-module.

Set $J = p\mathbb{Z} + \Delta(H)$. Then $\mathbb{Z}H/J \cong \mathbb{Z}/p\mathbb{Z}$, hence J is a (completely) prime ideal of $\mathbb{Z}H$. Since $P/P\Delta(H)$ is finitely generated, P/PJ is a finite dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$ and hence P/PJ is finite.

Let Σ denote the set of all square $\mathbb{Z}H$-matrices that are invertible modulo J. By Fact 7.1, the universal localization $\mathbb{Z}H_\Sigma$ is a local ring such that $\mathbb{Z}H_\Sigma / \operatorname{Jac}(\mathbb{Z}H_\Sigma) \cong \mathbb{Z}/p\mathbb{Z}$.

Now $P_\Sigma = P \otimes_{\mathbb{Z}H} \mathbb{Z}H_\Sigma$ is a projective module over the local ring $\mathbb{Z}H_\Sigma$, hence P_Σ is free by Fact 2.6. Also the image of P in P_Σ generates P_Σ as a $\mathbb{Z}H_\Sigma$-module. Since P/PJ is finite and the image of PJ in $P_\Sigma/P_\Sigma \operatorname{Jac}(P_\Sigma)$ is 0 (because the image of J in $\mathbb{Z}H_\Sigma / \operatorname{Jac}(Z H_\Sigma)$ is 0; cf. Fact 7.1), we deduce that $P_\Sigma/P_\Sigma \operatorname{Jac}(P_\Sigma)$ is a finitely generated $\mathbb{Z}H_\Sigma$-module. Therefore P_Σ is a finitely generated free $\mathbb{Z}H_\Sigma$-module.

Below we prove that the natural morphism $\mathbb{Z}H \to \mathbb{Z}H_\Sigma$ is an embedding. Then Fact 2.7 implies that P is a finitely generated $\mathbb{Z}H$-module, the desired contradiction.

By Fact 7.2, $\mathbb{Z}H$ is an Ore domain. Let F be the classical ring of quotients of $\mathbb{Z}H$ which is a skew field. Since $\mathbb{Z}H$ is an Ore domain, we can identify it with the universal localization with respect to the set Σ' of all square $\mathbb{Z}H$-matrices which are left (=right) regular, that is, whose images in F are invertible.

We show that $\Sigma \subseteq \Sigma'$. Suppose that A is an $n \times n$ matrix in Σ. Then left multiplication by A induces a morphism of free $\mathbb{Z}H$-modules $f_A \colon \mathbb{Z}H^n \to \mathbb{Z}H^n$ that is a monomorphism modulo J. Clearly this map is a monomorphism modulo $\Delta(H)$ (since every square $\mathbb{Z}$-matrix is regular, if it is regular modulo $p\mathbb{Z}$). Then, by Lemma 7.3, f_A is a monomorphism, hence $A \in \Sigma'$.

Thus $\Sigma \subseteq \Sigma'$, hence, by the universal property, there exists a morphism of rings $\mathbb{Z}H_\Sigma \to \mathbb{Z}H_{\Sigma'} = F$ completing the following commutative diagram:

$$
\begin{array}{ccc}
 & & \mathbb{Z}H_\Sigma \\
 & \nearrow & \vdots \\
\mathbb{Z}H & & \downarrow \\
 & \searrow & \\
 & & \mathbb{Z}H_{\Sigma'}
\end{array}
$$

Since $\mathbb{Z}H$ is a subring of F, the morphism $\mathbb{Z}H \to \mathbb{Z}H_\Sigma$ is an embedding, as desired.

We wonder whether the results of this paper can be extended from polycyclic-by-finite to arbitrary soluble groups.

Problem 7.5. *Is every non-finitely generated projective module over the integral group ring of a soluble group free?*

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